Mode Propagation in Nonuniform Circular Ducts with Potential Flow

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A previously reported closed-form solution is expanded to determine effects of isentropic mean flow on mode propagation in a slowly converging-diverging duct: a circular cosh duct. On the assumption of uniform steady fluid density, the mean flow increases the power transmission coefficient. The increase is related directly to the increase of the cutoff ratio at the duct throat. With the negligible transverse gradients of the steady fluid variables, the conversion from one mode to another is negligible, and the power transmission coefficient remains unchanged with the mean flow direction reversed.

Nomenclature

$A(z),\tilde{A}(z)$ = functions, Eqs. (61) and (83)
a = effective length of converging-diverging section
$B_M(x),B_0(x)$ = functions determining duct shape with or without mean flow, Eqs. (45) and (48)
$b(x)$ = duct radius varying along duct
b_0 = duct radius at throat ($x=0$)
b_- = radius of left-hand side uniform duct (inlet)
b_+ = radius of right-hand side uniform duct (exit)
c = constant sound speed in isochoric flow
c_0 = sound speed in still fluid
D_1,D_2,D_r,D_t = constants
F = hypergeometric function
h = radial variable in cylindrical coordinate system
I = acoustic power flux
J_m = Bessel function of order m
K = total acoustic power across duct
k = free space wavenumber ω/c_0
k_- or $k_{mn}^{(-)}$ = propagation constant of a mode or (m,n) mode in inlet
$k^+,k_{mn}^{(+)}$ = propagation constant of a mode or (m,n) mode in exit
$M(x)$ = Mach number of mean flow of constant steady density fluid
M_0 = Mach number at duct throat
M_- = Mach number of uniform flow in inlet
M_+ = Mach number of uniform flow in exit
m = circumferential mode numbers
n = radial mode numbers
$\hat{n}$ = unit vector normal to surface
P = steady pressure in flow
P_c = steady pressure in flow of constant density
P_δ = deviation $P-P_c$
p = acoustic pressure
Q_{mn} = normalized Bessel function corresponding to (m,n) mode, Eq. (72)
q = parameter, Eq. (56)
R = acoustic power reflection coefficient
$R(x)$ or $R(r)$ = axial wavefunction, Eqs. (30), (40), and (41)
r = radial variable in local spherical coordinate system ($dr \cong dx$)
$S(x,t)$ = interaction term, Eq. (17)
$S_\omega(x)$ = temporal Fourier transform of $S(x,t)$
T = acoustic power transmission coefficient
t = time
TL = acoustic power transmission loss, Eq. (92)
U = steady velocity of isentropic flow
U_c = steady velocity of constant density fluid flow
U_δ = deviation $U-U_c$
u = acoustic velocity field
v = constant, Eq. (57)
w = parameter, Eqs. (45) and (46)
w_0 = w with $M=0$
x = axial coordinate
$\mathbf{x}$ = three-dimensional coordinates
z = dimensionless axial coordinate, Eq. (55)
α or α_{mn} = eigenvalue corresponding to a duct mode or (m,n) mode
$\beta(x)$ = $b(x)/b_0$
$\beta_\pm$ = $b_\pm/b_0$
Γ = gamma function
γ = specific heat ratio
γ_0 = mode cut-off ratio at duct throat, Eq. (94)
γ_- = mode cut-off ratio at inlet, Eq. (93)
ϵ = parameter, 1 for $M=0$, Eq. (45)
ζ = frequency parameter, Eqs. (53) and (96)
η = normalized coordinate variable, Eq. (32)
θ = polar angle in local spherical coordinate system
θ_0 = half cone angle of duct segment, Fig. 1
μ = parameter, Eq. (47)
ν = frequency parameter, kb_0/α
ξ = see Eq. (12)
ρ = acoustic perturbation of density
ρ_0 = spatially varying steady density
ρ_c = constant steady density
ρ_δ = deviation $\rho_0-\rho_c$
σ = parameter, Eq. (66)
τ = b_+/b_-
$\Phi(x)$ = Fourier transform of acoustic velocity potential

Presented as Paper 82-0122 at the AIAA 20th Aerospace Sciences Meeting, Orlando, Fla., Jan. 11-14, 1982; submitted Jan. 22, 1982; revision received Aug. 23, 1982.

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ϕ = azimuthal angle variable
χ = see Eq. (13)
$\Psi(x,t)$ = acoustic velocity potential
ψ = axial wavefunction, Eq. (41)
ω = angular frequency

Introduction

IN a previously reported investigation of mode propagation in a nonuniform duct,[1] a closed-form solution was obtained for a particular class of converging-diverging ducts: *circular cosh ducts*. These ducts are composed of two asymptotically uniform ducts which are smoothly coupled through a converging-diverging section. The two uniform duct elements can differ from each other in cross-sectional area. The duct shape can be adjusted by means of three duct parameters to produce a variety of circular ducts of practical interest. The analysis, which was previously developed for the case of no mean flow, is here expanded to include mean flow effects.

The mean flow under consideration is subsonic, and is assumed to be isentropic. Viscous effects are excluded from discussion. Numerical results will be presented only for cases of low Mach numbers, for which the steady density and pressure hardly vary from one location to another in the duct. A systematic method was described elsewhere[2] to treat finite deviations of the steady fluid variables from the flow of constant density.

Wave Equation

As customary, the fluid variables are first decomposed into steady and fluctuating parts. The steady part will be further decomposed to treat its spatial variation. The steady density, pressure, and flow velocity are denoted, respectively, by ρ_0, P, and U; and the fluctuations by ρ, p, and u. Retaining only the first-order terms of the fluctuations, one obtains from the continuity equation

$$\left(\frac{\partial}{\partial t}+U\cdot\nabla\right)p+\gamma P\nabla\cdot u+\frac{\gamma P}{\rho_0}\nabla\rho_0\cdot u+(\nabla\cdot U)p=0 \tag{1}$$

and from the Euler's equation

$$\left(\frac{\partial}{\partial t}+U\cdot\nabla\right)u+(u\cdot\nabla)U+\frac{1}{\rho_0}\nabla p-\frac{\nabla P}{\gamma\rho_0 P}p=0 \tag{2}$$

Here γ is the specific heat ratio of the fluid, and we have used the adiabatic relation between density and pressure

$$p=(\gamma P/\rho_0)\rho \tag{3}$$

Also used are the zeroth-order relations

$$\nabla\cdot(\rho_0 U)=0 \tag{4}$$

$$\rho_0(U\cdot\nabla)U+\nabla P=0 \tag{5}$$

The steady variables are, in general, functions of spatial coordinates in a nonuniform duct with mean flow. The functions may be complicated. However, in a subsonic flow, the major portion of the steady variables may assume simple coordinate dependence, and the deviations from the simple dependence can be small. We now decompose the steady variables further as follows:

$$\rho_0=\rho_c+\rho_\delta \tag{6}$$

$$P=P_c+P_\delta \tag{7}$$

$$U=U_c+U_\delta \tag{8}$$

Here ρ_c, P_c, and U_c are a new set of steady variables which are simple functions of spatial coordinates, and ρ_δ, P_δ, and U_δ are the deviations. In the present analysis, ρ_c and P_c are chosen to be constant and set equal to the steady density and pressure in the left-side uniform duct element ($x<0$, or the inlet). The variable U_c is the velocity of the constant density flow and, therefore, it is subject to the isochoric flow condition

$$\nabla\cdot U_c=0 \tag{9}$$

On inserting Eqs. (6-8) into Eqs. (1) and (2), one obtains

$$\left(\frac{\partial}{\partial t}+U_c\cdot\nabla\right)p+\gamma P_c\nabla\cdot u+\xi=0 \tag{10}$$

$$\left(\frac{\partial}{\partial t}+U_c\cdot\nabla\right)u+(u\cdot\nabla)U_c+\frac{1}{\rho_c}\nabla p+\nabla\chi=0 \tag{11}$$

where

$$\xi=\nabla\cdot(pU_\delta)+\gamma P_\delta\nabla\cdot u+u\cdot\nabla P_\delta \tag{12}$$

$$\chi=U_\delta\cdot u-\frac{\rho_\delta}{\rho_c^2}\left(1-\frac{\rho_\delta}{\rho_c}\right)p+0(\rho_\delta^3 p) \tag{13}$$

With the potential flow condition $\nabla\times u=0$, one sets

$$u=\nabla\Psi(x,t) \tag{14}$$

where Ψ is the acoustic velocity potential. With the substitution of this equation, Eqs. (10) and (11) are combined to yield the equation

$$\nabla^2\Psi-\frac{1}{c^2}\left(\frac{\partial}{\partial t}+U_c\cdot\nabla\right)^2\Psi=S(x,t) \tag{15}$$

where

$$c^2\equiv\frac{\gamma P_c}{\rho_c}=c_0^2-\frac{\gamma-1}{2}U_c^2\qquad(x=-\infty) \tag{16}$$

$$S=\frac{1}{c^2}\left\{\left(\frac{\partial}{\partial t}+U_c\cdot\nabla\right)\chi-\frac{\xi}{\rho_c}\right\} \tag{17}$$

Equation (15) is, strictly speaking, a homogeneous equation for the acoustic field. It is, however, convenient to treat it as an inhomogeneous wave equation with the interaction term S. With $S=0$, Eq. (15) is the homogeneous wave equation in a mean flow of constant density and pressure. The term S represents the interaction between the acoustic field and the mean flowfield due to the spatial variations ρ_δ, P_δ, and U_δ of the latter. Since no acoustic energy is created or annihilated, the interaction accounts for the elastic scattering of sound by the spatial nonuniformity of the steady fluid variables.

Equation (15) is linear for the acoustic field. Thus, the temporal Fourier transform yields

$$\nabla^2\Phi(x)+(1/c^2)(\omega+iU_c\cdot\nabla)^2\Phi(x)=S_\omega(x) \tag{18}$$

Here

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,t)e^{i\omega t}\mathrm{d}t \tag{19}$$

$$S_\omega(x)=\frac{1}{c^2}\left\{(-i\omega+U_c\cdot\nabla)\chi_\omega(x)+\frac{1}{\rho_c}\xi_\omega(x)\right\} \tag{20}$$

with

$$\chi_\omega(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\chi(x,t)e^{i\omega t}\mathrm{d}t \tag{21}$$

$$\xi_\omega(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \xi(x,t) e^{i\omega t} \mathrm{d}t \tag{22}$$

Homogeneous Solutions

We consider solutions to Eq. (18) first with the interaction term neglected. The wave equation is then

$$\nabla^2 \Phi_H + (1/c^2)(\omega + iU_c \cdot \nabla)^2 \Phi_H = 0 \tag{23}$$

where the subscript H has been used to indicate the homogeneous solutions. The homogeneous solutions serve two purposes. First, with boundary conditions properly satisfied, they describe mode propagation in the duct with mean flow of uniform density, which can be described adequately by the variables ρ_c, P_c, and U_c. It is likely that potential flow solutions will be close to these variables for low Mach numbers. Second, the homogeneous solutions can be used as basis functions to construct the Green's function. With the help of the latter, Eq. (18) can be systematically solved by means of the iteration-perturbation method for cases when the deviations of steady fluid variables are finite but small.[3]

Let us consider Eq. (23) in a circular duct which is specified in terms of the radius $b(x)$, x being the axial coordinate variable. As in Ref. 1, we consider first a small duct segment and the local spherical coordinates (r,θ,ϕ) as illustrated in Fig. 1. The azimuth ϕ, the angle variable around the duct axis, is not shown in the figure. The duct segment is so short that it may be regarded locally conical. Thus the slope, $b = \mathrm{d}b/\mathrm{d}x$, of the duct wall remains constant within the segment. The origin of the local spherical coordinate system is located at a point on the duct axis such that the coordinate surface $\theta = \theta_0$ (cone) tangentially contacts the duct wall in the segment. The half cone angle θ_0 is related to the wall slope b' as

$$b' = \tan\theta_0 \tag{24}$$

With the spherical coordinates, the isochoric flow velocity is chosen as

$$U_c = \hat{r}(\Omega/r^2) \tag{25}$$

where Ω is a constant with the sign being opposite for converging and diverging sections, and $\hat{r}$ is the radial unit vector. Equation (23) can then be written for the spherical coordinates as

$$\frac{1}{r^2}\left\{\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\Phi_H + \frac{1}{c^2}\left(\omega + \frac{i\Omega}{r^2}\frac{\partial}{\partial r}\right)^2 \Phi_H = 0 \tag{26}$$

This equation can be separated into three ordinary differential equations. With the substitution

$$\Phi_H(x) = R(r)J(\theta)H(\phi) \tag{27}$$

one obtains, from Eq. (26)

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2} + m^2\right)H = 0 \tag{28}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\frac{\mathrm{d}J}{\mathrm{d}\theta} + \left(\frac{\alpha^2}{\sin^2\theta_0} - \frac{m^2}{\sin^2\theta}\right)J = 0 \tag{29}$$

$$(1-M^2)\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \left[\frac{2(1+M^2)}{r} - 2ikM\right]\frac{\mathrm{d}R}{\mathrm{d}r} + \left[k^2 - \left(\frac{\alpha}{r\sin\theta_0}\right)^2\right]R = 0 \tag{30}$$

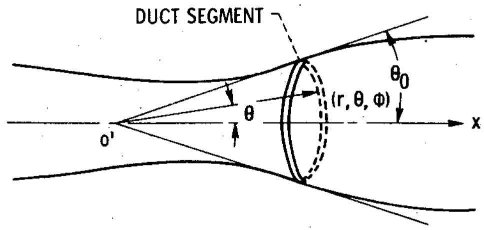

Fig. 1 Local spherical coordinate system for a segment of a converging-diverging duct.

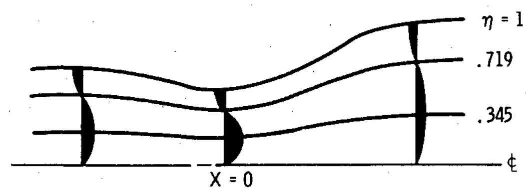

Fig. 2 Coordinate surfaces ($\eta = 0.345, 0.719, 1$) and transverse shape of (1,1) mode at various axial locations.

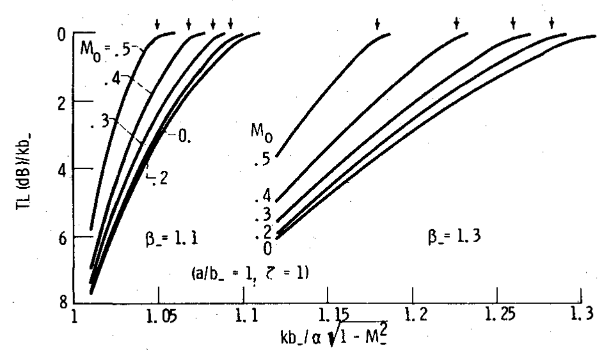

Fig. 3 Power transmission loss (TL) of (8,5) mode, divided by kb_-, vs inlet cutoff ratio (γ_-).

where m and α are the separation constants, and

$$k = \frac{\omega}{c}, \qquad M = \frac{\Omega}{cr^2}$$

The solution to Eq. (28) is

$$H = e^{im\phi} \tag{31}$$

with m being an integer. As for Eq. (29), it is convenient to introduce the normalized coordinate variable η, defined as

$$\eta = \frac{\sin\theta}{\sin\theta_0} \quad \text{for } \theta_0 \neq 0$$
$$= \frac{h}{b} \quad \text{for } \theta_0 = 0 \tag{32}$$

where h is the radial variable in the cylindrical coordinate system. The value of η varies from 0 to 1 in the duct. On the duct axis, $\eta = 0$; and on the duct wall, $\eta = 1$. Equation (29) is written in terms of η as

$$(1 - \sin^2\theta_0\eta^2)\frac{\mathrm{d}^2J}{\mathrm{d}\eta^2} + \frac{1}{\eta}(1 - 2\sin^2\theta_0\eta^2)\frac{\mathrm{d}J}{\mathrm{d}\eta} + \left(\alpha^2 - \frac{m^2}{\eta^2}\right)J = 0 \tag{33}$$

For a duct of slowly varying cross section ($b' \ll 1$), $\sin\theta_0 \cong b'$. Neglecting terms involving $(b')^2$ and higher orders, one can write Eq. (33) as

$$\frac{d^2J}{d\eta^2}+\frac{1}{\eta}\frac{dJ}{d\eta}+\left(\alpha^2-\frac{m^2}{\eta^2}\right)J=0 \tag{34}$$

This is the Bessel equation of order m, and the physically acceptable solution is

$$J\sim J_m(\alpha\eta) \tag{35}$$

where J_m is the Bessel function of order m. From the boundary condition of the hard duct wall, one determines values of α as follows:

$$\alpha=\alpha_{mn}, \qquad n=0,1,2... \tag{36}$$

where α_{mn} is the nth zero of $dJ_m(x)/dx$.

Some remarks should be made on the coordinate variable η. Unlike the variable θ, η is not a local coordinate variable, but it is valid through the entire duct. To see this, let us consider two small duct segments 1 and 2, respectively, at $x=x_1$ and x_2. The half cone angles of the respective segments are denoted by $\theta_{0,1}$ and $\theta_{0,2}$. Assume $\theta_{0,1}>\theta_{0,2}$. The variable θ varies from 0 to $\theta_{0,1}$ in segment 1, and from 0 to $\theta_{0,2}$ in segment 2. A value of θ greater than $\theta_{0,2}$ and less than $\theta_{0,1}$ represents a coordinate surface in segment 1, but not in segment 2. Furthermore, two coordinate surfaces represented by a value of θ less than $\theta_{0,2}$, respectively in the two duct segments, in general, are not equivalent to each other. On the other hand, η varies from 0 to 1 for all the duct segments. The locations represented by a value of η are in one-to-one correspondence for two different duct segments. A coordinate surface is uniquely determined as the locus of $\eta=\eta_1$, η_1 being a constant, and the coordinate surface spans all the duct segments. A coordinate surface with a value of η does not intersect with any other coordinate surface with a different value of η. The duct wall is a coordinate surface corresponding to $\eta=1$. We also note, from Eqs. (35) and (36), that an eigenfunction assumes the same value on a coordinate surface $\eta=\eta_1$. A mode is in one-to-one correspondence with an eigenfunction. Thus, with the use of the variable η, a mode is defined in a nonuniform circular duct with isochoric flow if b' remains small (Fig. 2).

Equation (30) governs the propagation of a mode along the duct. The equation is first transformed from the local coordinate r to the duct coordinate x. To this end we use the relations

$$\sin\theta_0=b' \tag{37}$$

$$r=b/b' \tag{38}$$

$$\frac{d}{dr}=\frac{d}{dx} \tag{39}$$

These relations are valid for all the duct segments and the error is of order $(b')^2$. In order to avoid misunderstandings, a remark should be made on Eq. (39). Equation (30) governs the wave along a line that is the intersection of coordinate surfaces $\phi=$const and $\eta=$const (not $\theta=$const). On this line, $dr=dx\sqrt{1+b'^2}$, and Eq. (39) follows. One should not attempt to derive a similar relation by regarding r as a function of h as well as x. Such a derivation might lead one to an incorrect conclusion, and the error would appear to be of order b' instead of $(b')^2$. Note that the origin of the local coordinate system changes with x. It cannot be overemphasized that R [as given in Eq. (30)] deals with the wavefield variation along a line which is the intersection of coordinate surfaces $\phi=$const and $\eta=$const. Finally, the use of the relations of Eqs. (37-39) is very unlikely to lead one to any asymptotic paradox.

Using Eqs. (37-39), one can write Eq. (30) as

$$(1-M^2)\frac{d^2R}{dx^2}+\left[\frac{2b'}{b}(1+M^2)+2ikM\right]\frac{dR}{dx}+\left(k^2-\frac{\alpha^2}{b^2}\right)R=0 \tag{40}$$

Here we have used

$$M(x)=M_0/\beta^2(x), \qquad \beta(x)=b(x)/b_0$$

where b_0 and M_0 are the duct radius and the Mach number at the duct throat. For a given $\beta(x)$, $M(x)$ is determined once M_0 is known. Thus, for a given duct shape, M_0 is sufficient to specify $M(x)$.

With the substitution

$$R(x)=\frac{\psi(x)}{b(x)\sqrt{1-M^2(x)}}\exp\left(-ik\int^x\frac{M(x)}{1-M^2(x)}dx\right) \tag{41}$$

Eq. (40) is transformed into

$$\frac{d^2\psi}{dx^2}+\left[\left(\frac{k}{1-M^2}\right)^2-\frac{1}{1-M^2}\frac{\alpha^2}{b^2}\right]\psi=0 \tag{42}$$

where terms like b'^2 and b'' have been neglected.

Solutions in a Circular Cosh Duct

Equation (42) was solved exactly in a closed form with a circular cosh duct when $M=0$, in Ref. 1. With $M\neq0$, it is convenient to introduce a function $B_M(x)$ as follows:

$$B_M(x)=\frac{1}{(1-M^2)\beta^2}-\nu^2\frac{M^2(2-M^2)}{(1-M^2)^2} \tag{43}$$

where $\nu=kb_0/\alpha$. Note that this function depends on the Mach number and the frequency (ν) as well as the duct shape ($\beta(x)$). With the use of Eq. (43), Eq. (42) is written as

$$\frac{d^2\psi}{dx^2}+\left[k^2-\frac{\alpha^2}{b_0^2}B_M(x)\right]\psi=0 \tag{44}$$

The discussions thus far apply to any nonuniform hardwall circular duct as long as b'^2 and b'' are negligible.

To utilize the cosh duct solution, we set

$$B_M(x)=B_M(0)+w\{\cosh^2\mu\ \operatorname{sech}^2(\epsilon x/a-\mu)-\sinh(2\mu)\tanh(\epsilon x/a-\mu)-\cosh(2\mu)\} \tag{45}$$

Here a is the effective length of the converging-diverging section, and w, μ, and ϵ are parameters depending on M, ν, and duct geometry. The parameters w and μ are determined as

$$w=\{[B_M(0)-B_M(\infty)][B_M(0)-B_M(-\infty)]\}^{1/2} \tag{46}$$

$$\mu=\frac{1}{4}\ln\left\{\frac{B_M(0)-B_M(+\infty)}{B_M(0)-B_M(-\infty)}\right\} \tag{47}$$

Involved in these equations are M, ν, and $\beta_\pm$, the radius ratios of the duct. For fixed values of M_0 and ν, w and μ are determined from the radius ratios $\beta_\pm$, and vice versa. One notices from Eq. (47) that μ is equal to zero for a symmetric duct ($\beta_+=\beta_-$), regardless of values of M_0 and ν.

The parameter ϵ is not given in a closed form, but its value can be determined analytically. For fixed values of M_0 and ν, the duct radius ratios $\beta_\pm$ have been used for the determination of w and μ. The axial variation of the radius in the

Table 1 Value of ϵ

β_-	M_0	Eigenvalue	ζ = −0.4	−0.2	0	0.2
1.1	0.2	4.20		1.0020	1.0014	1.0007
		8.54	1.0020	1.0017	↓	1.0010
		16.53	1.0017	1.0015	↓	1.0012
		23.27	1.0016	1.0015	↓	1.0012
	0.4	4.20			0.9920	0.9845
		8.54	0.9980	0.9952	↓	0.9886
		16.53	0.9953	0.9937	↓	0.9903
		23.27	0.9944	0.9932	↓	0.9908
1.3	0.2	4.20	1.0039	1.0026	1.0012	0.9997
		8.54	1.0025	1.0019	↓	1.0004
		16.53	1.0019	1.0015	↓	1.0008
		23.27	1.0017	1.0014	↓	1.0009
	0.4	4.20	1.0041	0.9948	0.9839	0.9710
		8.54	0.9946	0.9895	↓	0.9778
		16.53	0.9897	0.9868	↓	0.9808
		23.27	0.9880	0.9860	↓	0.9817

Table 2 Value of ϵ at $\zeta=0$

M_0	β_- = 1.1	1.2	1.3	1.4	1.5
0.1	1.0004	1.0006	1.0005	1.0004	1.0002
0.2	1.0014	1.0015	1.0012	1.0005	0.9998
0.3	1.0008	1.0001	0.9987	0.9971	0.9954
0.4	0.9920	0.9875	0.9837	0.9806	0.9776
0.5	0.9438	0.9316	0.9271	0.9244	0.9223

converging-diverging section will be used to determine ϵ. If the value of ϵ were kept constant, the duct shape would depend on M and ν as can be seen readily from Eqs. (43) and (45). Thus, the value of ϵ must be so determined that the duct shape may remain unchanged for different values of M and ν. To this end, at a properly chosen axial location in the converging-diverging section, the duct radius is required to be the same for all the allowed values of M_0 and ν. We choose $M_0=0$ as the reference case. With $M_0=0$, we set $\epsilon=1$, and obtain from Eq. (43)

$$B_0(x)\equiv B_M(x)\,|_{M=0}=\beta^{-2}(x) \tag{48}$$

and from Eq. (45)

$$B_0(x)=1+w_0\{\cosh^2\mu_0\operatorname{sech}^2(x/a-\mu_0)$$
$$-\sinh(2\mu_0)\tanh(x/a-\mu_0)-\cosh(2\mu_0)\} \tag{49}$$

where the subscript 0 indicates $M=0$. Note that B_0, w_0, and μ_0 do not depend on ν. The axial location is chosen preferably in the converging section, as $x=-s$. The duct radius at that location is determined from Eqs. (48) and (49).

$$\beta(-s)=\{1+w_0[\cosh^2\mu_0\operatorname{sech}^2(s/a+\mu_0)$$
$$+\sinh(2\mu_0)\tanh(s/a+\mu_0)-\cosh(2\mu_0)]\}^{-1/2} \tag{50}$$

With $M_0\neq 0$, Eq. (43) is written for $x=-s$ as

$$B_M(-s)=\frac{1}{[1-M^2(-s)]\beta^2(-s)}$$
$$-\nu^2\frac{M^2(-s)[2-M^2(-s)]}{[1-M^2(-s)]^2} \tag{51}$$

and Eq. (45) is

$$B_M(-s)=B_M(0)+w\{\cosh^2\mu\ \operatorname{sech}^2(\epsilon s/a+\mu)$$
$$+\sinh(2\mu)\tanh(\epsilon s/a+\mu)-\cosh(2\mu)\} \tag{52}$$

The value of ϵ is determined from Eqs. (50-52). They may exist more than one real solution for ϵ; the correct one should be determined from physical considerations. A list of values of ϵ are given in Table 1, for various values of M_0 and ζ, where ζ is a frequency parameter defined as

$$\zeta=\frac{1}{1-M_0^2}(kb_0-\alpha\sqrt{1-M_0^2}) \tag{53}$$

As the table shows, ϵ is close to unity in most cases of interest. Also notice that ϵ takes the same value for all the modes at $\zeta=0$. The values of ϵ at $\zeta=0$ are listed for various values of M_0 and β_- in Table 2.

We now return to Eq. (44). On inserting Eq. (45) into Eq. (44) we obtain

$$\frac{d^2\psi}{dz^2}+[q^2-v(\cosh^2\mu\ \operatorname{sech}^2 z-\sinh(2\mu)\tanh z)]\psi=0 \tag{54}$$

where

$$z=\epsilon x/a-\mu \tag{55}$$

$$q^2=(ka/\epsilon)^2-(\alpha a/\epsilon b_0)^2[B_M(0)-w\ \cosh(2\mu)] \tag{56}$$

$$v=(\alpha a/\epsilon b_0)^2 w \tag{57}$$

Equation (54) can be solved exactly.[3] The general solution is written as

$$\psi=D_1\psi_1(z)+D_2\psi_2(z) \tag{58}$$

Here D_1 and D_2 are constants; and ψ_1 and ψ_2 are linearly independent solutions.

$$\psi_1=A(z)F(\zeta_1,\zeta_2,\zeta_3;\ (1+e^{2z})^{-1}) \tag{59}$$

$$\psi_2=A(z)(1+e^{2z})^{-ik+a/\epsilon}$$
$$\times F(\zeta_2-\zeta_3+1,\ \zeta_1-\zeta_3+1,\ 2-\zeta_3;\ (1+e^{2z})^{-1}) \tag{60}$$

where F is the hypergeometric function, and

$$A(z) = e^{i(k_+ - k_-)za/2\epsilon}(2\cosh z)^{i(k_+ + k_-)a/2\epsilon} \quad (61)$$

$$k_\pm = \frac{k}{1-M_\pm^2}\left[1-(1-M_\pm^2)\left(\frac{\alpha}{kb_\pm}\right)^2\right]^{1/2} \quad (62)$$

$$\zeta_1 = \{1 - i[(k_+ + k_-)a/\epsilon + \sigma]\}/2 \quad (63)$$

$$\zeta_2 = \{1 - i[(k_+ + k_-)a/\epsilon - \sigma]\}/2 \quad (64)$$

$$\zeta_3 = 1 - ik_+ a/\epsilon \quad (65)$$

$$\sigma = (4v \ \cosh^2\mu - 1)^{1/2} \quad (66)$$

The constants D_1 and D_2 are to be determined from boundary conditions at $x = \pm\infty$. The asymptotic forms of the hypergeometric function are used to find:

1) at $x = +\infty$

$$\psi_1 = e^{-ik_+\mu a/\epsilon} e^{ik_+ x} \quad (67)$$

$$\psi_2 = e^{ik_+\mu a/\epsilon} e^{-ik_+ x} \quad (68)$$

2) at $x = -\infty$

$$\psi_1 = \frac{\Gamma(\zeta_3)\Gamma(\zeta_1+\zeta_2-\zeta_3)}{\Gamma(\zeta_1)\Gamma(\zeta_2)} e^{-ik_-\mu a/\epsilon + ik_- x} + \frac{\Gamma(\zeta_3)\Gamma(\zeta_3-\zeta_1-\zeta_2)}{\Gamma(\zeta_3-\zeta_1)\Gamma(\zeta_3-\zeta_2)} e^{ik_-\mu a/\epsilon - ik_- x} \quad (69)$$

$$\psi_2 = \frac{\Gamma(2-\zeta_3)\Gamma(\zeta_1+\zeta_2-\zeta_3)}{\Gamma(\zeta_2-\zeta_3+1)\Gamma(\zeta_1-\zeta_3+1)} e^{-ik_-\mu a/\epsilon + ik_- x} + \frac{\Gamma(2-\zeta_3)\Gamma(\zeta_3-\zeta_1-\zeta_2)}{\Gamma(1-\zeta_1)\Gamma(1-\zeta_2)} e^{ik_-\mu a/\epsilon - ik_- x} \quad (70)$$

where Γ is the gamma function.

Mode Propagation in Circular Cosh Ducts with Low Mach Number Flow

As mentioned earlier, the deviations ρ_δ, P_δ, and U_δ are negligible in cases of low Mach numbers ($M_0 < 0.5$). In these cases, the solutions can be obtained as in Eq. (58), with the boundary conditions to be satisfied. Note that these solutions are the zeroth-order solutions to Eq. (18) and can be used to obtain more accurate solutions.

Let us begin with the incident wave that is composed of the (m,n) mode coming from $x = -\infty$. In the incident side uniform duct,

$$\Phi_1 = Q_{mn}(\eta)e^{im\phi}\frac{1}{b_-\sqrt{1-M_-^2}} \times \exp[i(-kM_-/(1-M_-^2) + k_{mn}^{(-)})x] \quad (71)$$

where

$$Q_{mn}(\eta) = J_m(\alpha_{mn}\eta)/L_{mn} \quad (72)$$

$$L_{mn} = \left\{\int_0^1 J_m^2(\alpha_{mn}\eta)\eta \mathrm{d}\eta\right\}^{1/2} \quad (73)$$

$$k_{mn}^{(\pm)} = k_\pm |_{\alpha=\alpha_{mn}} \quad (74)$$

Correspondingly, we have

$$\psi_I = e^{ik_{mn}^{(-)}x} \quad (75)$$

With mode conversion neglected, the solution ψ is given at $x = \pm\infty$ in the form

$$\lim_{x\to-\infty} \psi(x) = e^{ik_{mn}^{(-)}x} + D_r e^{-ik_{mn}^{(-)}x} \quad (76)$$

$$\lim_{x\to\infty} \psi(x) = D_t e^{ik_{mn}^{(+)}x} \quad (77)$$

The second term of Eq. (76) represents the reflected wave, and Eq. (77) the transmitted wave. The constants in Eq. (58) are determined readily by comparing the asymptotic expressions in Eqs. (67-70) with Eqs. (76) and (77).

$$D_2 = 0 \quad (78)$$

$$D_1 = e^{ik_{mn}^{(-)}\mu a/\epsilon}\frac{\Gamma(\zeta_1)\Gamma(\zeta_2)}{\Gamma(\zeta_3)\Gamma(\zeta_1+\zeta_2-\zeta_3)} \quad (79)$$

The coefficients D_r and D_t are

$$D_r = e^{2ik_{mn}^{(-)}\mu a/\epsilon}\frac{\Gamma(\zeta_1)\Gamma(\zeta_2)\Gamma(\zeta_3-\zeta_1-\zeta_2)}{\Gamma(\zeta_1+\zeta_2-\zeta_3)\Gamma(\zeta_3-\zeta_1)\Gamma(\zeta_3-\zeta_2)} \quad (80)$$

$$D_t = e^{i(k_{mn}^{(-)} - k_{mn}^{(+)})\mu a/\epsilon}\frac{\Gamma(\zeta_1)\Gamma(\zeta_2)}{\Gamma(\zeta_3)\Gamma(\zeta_1+\zeta_2-\zeta_3)} \quad (81)$$

In Eqs. (78-81) the parameters μ, ϵ, ζ_1, ζ_2, and ζ_3 contain the constant α; it is to be replaced by the eigenvalue α_{mn}. The solution is

$$\Phi = Q_{mn}(\eta)e^{im\phi}b^{-1}(1-M^2)^{-1/2}\exp\left(ik\int_0^x \frac{M}{1-M^2}\mathrm{d}x\right) \times \tilde{A}(z)\Gamma(\zeta_1,\zeta_2,\zeta_3;(1+e^{2z})^{-1}) \quad (82)$$

where

$$\tilde{A}(z) = D_1 A(z)\exp\left(i\frac{kM_-}{1-M^2}x_- + ik\int_{x_-}^0 \frac{M}{1-M^2}\mathrm{d}x\right) \quad (83)$$

x_- being a large negative value of x. Recall that $\eta = h/b(x)$ and $z = \epsilon x/a - \mu$.

Power Reflection and Transmission Coefficients

The acoustic power intensity in the isentropic flow is given by[4]

$$I = \langle pu\rangle + U(\langle p^2\rangle/\rho_0 c^2 + U\cdot\langle pu\rangle/c) + \rho_0\langle(U\cdot u)u\rangle \quad (84)$$

where $\langle\ \rangle$ stands for the time average. The total power across the duct is obtained as

$$K = \int I\cdot\hat{n}\mathrm{d}A \quad (85)$$

where $\mathrm{d}A$ is the surface area element and $\hat{n}$ the unit vector normal to the surface. In the uniform duct element containing the axisymmetric mean flow, we have

$$K = 2\pi\int I_x h\mathrm{d}h \quad (86)$$

with the axial component I_x of the power intensity given by

$$I_x = \langle pu_x\rangle(1+M^2) + \rho_0 U(\langle u_x^2\rangle + \langle p^2\rangle\rho_0^2 c^2) \quad (87)$$

Let K_i, K_r, and K_t denote the total powers, respectively, of the incident, the reflected, and the transmitted wave. The power reflection and transmission coefficients, R and T, are defined as

$$R = K_r/K_i \quad (88)$$

$$T = K_t / K_i \tag{89}$$

Using Eqs. (71), (81), (76), and (77), one can obtain

$$R = \frac{\cosh[\pi(k_+ - k_-)a/\epsilon] + \cosh(\pi\sigma)}{\cosh[\pi(k_+ + k_-)a/\epsilon] + \cosh(\pi\sigma)} \tag{90}$$

$$T = \frac{2\sinh(\pi k_+ a/\epsilon)\cdot\sinh(\pi k_- a/\epsilon)}{\cosh[\pi(k_+ + k_-)a/\epsilon] + \cosh(\pi\sigma)} \tag{91}$$

Note that $R+T=1$. That is, the acoustic energy is conserved as it should be in a linear analysis of the acoustic field. The terms neglected in the approximation would not create or annihilate the acoustic energy. They, if included, might transfer the energy between modes or between the reflected and the transmitted waves. Also note that Eqs. (90) and (91) are similar to Eqs. (25) and (26) in Ref. 1. The mean flow dependence is included only through the parameters $k_\pm$, ϵ, and σ. The latter are even functions of M. Thus, the power reflection and transmission coefficients are independent of the mean flow direction, the positive or the negative x direction.

Numerical results will be discussed in terms of the acoustic power transmission loss (TL) in the circular cosh ducts. The mean flow is isentropic flow and has a uniform steady fluid density. As mentioned earlier, a uniform steady density is a good approximation for low Mach number flow in inlet ducts. Numerical calculation includes the Mach numbers up to the value of 0.5 at the duct throat.

The TL is defined as

$$\mathrm{TL} = -10\log_{10}(T) \tag{92}$$

For the present calculation, Eq. (91) is used for T. The TL is plotted as a function of the throat Mach number M_0 or of the frequency parameter γ_- or ζ. The parameter γ_- is the mode cutoff ratio referenced to the left-hand side uniform duct element, and is referred to as the inlet cutoff ratio.

$$\gamma_- = kb_- / \alpha\sqrt{1-M_-^2} \tag{93}$$

The inlet cutoff ratio is a convenient parameter characterizing the modes contained in the incident wave. The sound generated in a fan duct often comprises many different modes. In such a case, the mode distribution can be obtained as a function of the cutoff ratio. Another cutoff ratio useful for discussion is that referenced to the duct throat, given by

$$\gamma_0 = kb_0 / \alpha\sqrt{1-M_0^2} \tag{94}$$

If $\gamma_0 > 1$, the mode is cut on through the entire duct. On the other hand, if $\gamma_0 < 1$, the mode that is cut on initially ($\gamma_- > 1$), is cut off in the converging-diverging section. Except for low frequencies ($kb_0 < 3$), the TL is approximately 3 dB for $\gamma_0 = 1$, independent of the duct geometry or the Mach number. The two cutoff ratios are related to each other as

$$\gamma_- = \gamma_0 \beta_- [(1-M_0^2)/(1-M_0^2\beta_-^4)]^{1/2} \tag{95}$$

The parameter ζ is defined in Eq. (53), and can be written as

$$\zeta = \frac{kb_0}{1-M_0^2}\left(1-\frac{1}{\gamma_0}\right) \tag{96}$$

This parameter collapses the TL curves for many different modes.

In Fig. 3, the TL divided by kb_- is plotted as a function of γ_- for the various values of M_0 between 0 and 0.5, and for $\beta_- = 1.1$ or 1.3. The other duct parameters are $a/b_- = 1$, $\zeta = 1$. On the top of the figure are shown the arrow marks, each pointing to a value of γ_- which corresponds to $\gamma_0 = 1$ with the given value of M_0 [cf. Eq. (95)]. The (8,5) mode has

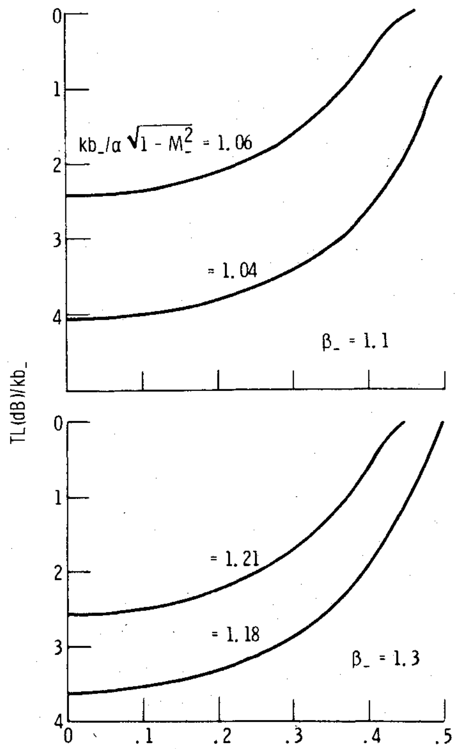

Fig. 4 Power transmission loss (TL) of (8,5) mode, divided by kb_-, vs Mach number at throat (M_0).

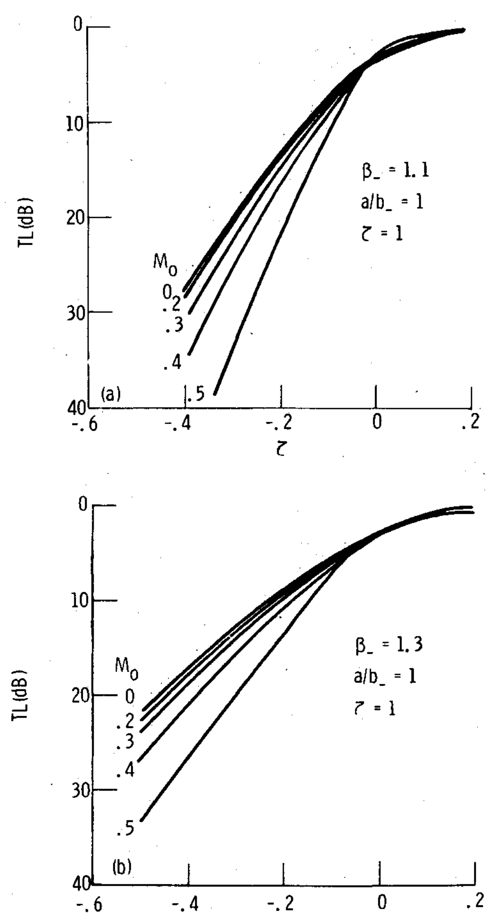

Fig. 5 Power transmission loss (TL) vs ζ. Each curve representing modes with eigenvalue greater than 3.

been used for the calculation. However, except in the vicinity of $\gamma_0 = 1$, the results may be used for modes with eigenvalues from 8 to 25, within an error of 1 dB. As expected, the TL decreases with the increasing value of γ_- (see Ref. 1). The decreasing rate of the TL is faster for larger Mach number. For a fixed value of γ_-, the TL is smaller for larger Mach number. Since TL/kb_- is plotted in Fig. 3, the TL itself is proportional to kb_- for a fixed value of γ_-. It follows that, for a mode distribution given as a function of the inlet cutoff ratio, the converging-diverging duct contour reduces the transmitted sound more effectively for cases when the distribution comprises modes of larger eigenvalues.

In Fig. 4, the TL/kb_- is plotted as a function of M_0 for fixed values of γ_-. As M_0 increases, the TL decreases very slowly in the beginning ($M_0 \cong 0$), and then decreases at a gradually increasing rate until M_0 reaches a value for which $\gamma_0 = 1$. This mean flow dependence of the TL is directly related to the change of the cutoff ratio at the throat. As one can notice from Eq. (95), γ_0 increases with increasing M_0 for γ_- held constant.

In Fig. 5, the TL is plotted as a function of ζ. Each curve, for a value of M_0, includes many modes with eigenvalues greater than 5. The TL increases with increasing ζ, and becomes approximately 3 dB at $\zeta = 0$ independent of the Mach number and the duct parameters. The advantage of this presentation is that almost all the modes are subject to the same TL for a given value of ζ. The mean flow dependence of the TL in this figure appears misleading. However, note that, with M_0 increased, γ_0 increases and kb_0 should be decreased to keep ζ unchanged [see Eq. (96)].

Concluding Remarks

In an attempt to improve the understanding of the acoustic characteristics of a fan duct system, mode propagation has been investigated in a particular class of converging-diverging circular ducts: *circular cosh ducts,* with isentropic and inviscid mean flow. The duct shape can be adjusted by means of three duct parameters, and covers a wide range of converging-diverging ducts of practical interest.

On the assumption that the duct cross-sectional area varies slowly, an approximate wave equation has been derived. The equation is divided into two parts: 1) the homogeneous wave equation involving the mean flow of uniform steady density fluid, and 2) the interaction between the acoustic field and the deviation of the steady fluid variables from the constant steady density fluid flow. The homogeneous equation is solved in a closed form, and the interaction term can be treated by means of an iteration-perturbation method. The solution of the homogeneous equation is regarded as a good approximation in cases of low Mach number mean flow.

With the interaction neglected, a mode is preserved, and the acoustic power transmission coefficient increases with the mean flow increase. This mean flow effect is related directly to the change of the cutoff ratios. With the increasing mean flow, the cutoff ratio at the duct throat increases faster than at the uniform duct sections (inlet and exit). For given duct geometry and mean flow, the TL of many different modes can be made to collapse onto a single curve with a proper choice of frequency parameter (ζ). The mean flow effects remain unchanged with the flow reversed. The interaction term needs numerical computations for its detailed discussions. However, the formal solution shows that a mode may be converted to others only if the transverse gradients of the steady fluid variables are not negligible.[2,5]

Acknowledgment

The work contained herein was partially based on consulting reports to Pratt & Whitney Aircraft (Oct. 30, 1974 and May 20, 1975).

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